

# Stable reduction of three point covers

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This note gives a survey on some results related to the stable reduction of three point covers, which were the topic of my talk at the *Journées Arithmétiques* 2003 in Graz. I thank the organizers of this very nice conference for inviting me and letting me give a talk.

## 1 Three point covers

**1.1 Ramification in the field of moduli** Let  $X$  be a smooth projective curve over  $\mathbb{C}$ . A celebrated theorem of Belyi states that  $X$  can be defined over a number field  $K$  if and only if there exists a rational function  $f$  on  $X$  with exactly three critical values, see [2], [11]. If such a function  $f$  exists, we can normalize it in such a way that the critical values are  $0, 1$  and  $\infty$ . After this normalization, we may view  $f$  as a finite cover  $f : X \rightarrow \mathbb{P}^1$  which is étale over  $\mathbb{P}^1 - \{0, 1, \infty\}$ . We call  $f$  a *three point cover*. Another common name for  $f$  is *Belyi map*. The *monodromy group* of  $f$  is defined as the Galois group of the Galois closure of  $f$ .

Let  $f : X \rightarrow \mathbb{P}^1$  be a three point cover. By the ‘obvious direction’ of Belyi’s theorem,  $f$  can be defined over the field  $\bar{\mathbb{Q}}$  of algebraic numbers. Therefore, for every element  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  of the absolute Galois group of  $\mathbb{Q}$  we obtain a conjugate three point cover  $f^\sigma : X^\sigma \rightarrow \mathbb{P}^1$ , which may or may not be isomorphic to  $f$ . This yields a continuous action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on the set of isomorphism classes of three point covers. Hence we can associate to  $f$  the number field  $K$  such that  $\text{Gal}(\bar{\mathbb{Q}}/K)$  is precisely the stabilizer of the isomorphism class of  $f$ . The field  $K$  is called the *field of moduli* of  $f$ . Under certain extra assumptions on  $f$ , the field  $K$  is the smallest field of definition of the cover  $f$ , see [5].

Three point covers are determined, up to isomorphism, by finite, purely combinatorial data – e.g. by a *dessin d’enfants* [17]. It is an interesting problem to describe the field of moduli of a three point cover in terms of these data. There are only few results of a general nature on this problem. The aim of this note is to explain certain results leading to the following theorem, proved in [21].

**Theorem 1.1** Let  $f : X \rightarrow \mathbb{P}^1$  be a three point cover, with field of moduli  $K$  and monodromy group  $G$ . Let  $p$  be a prime number such that  $p^2$  does not divide the order of  $G$ . Then  $p$  is at most tamely ramified in the extension  $K/\mathbb{Q}$ .

If the prime  $p$  does not divide the order of  $G$  then  $p$  is even unramified in the extension  $K/\mathbb{Q}$ , by a well known theorem of Beckmann [1]. Both Beckmann's result and Theorem 1.1 rely on an analysis of the reduction of  $f$  at the prime ideals  $\mathfrak{p}$  of  $K$  dividing  $p$ . The results leading to Theorem 1.1 were mainly inspired by Raynaud's paper [16].

**1.2 Good reduction** For a discussion of the relation between the ramification in the extension  $K/\mathbb{Q}$  and the reduction behaviour of  $f$ , it is convenient to localize at a prime ideal  $\mathfrak{p}$  of  $K$  dividing  $p$ . In other words, we may replace the extension  $K/\mathbb{Q}$  by a finite extension of  $p$ -adic fields.

Fix a prime number  $p$  and let  $K_0$  denote the completion of the maximal unramified extension of  $\mathbb{Q}_p$ . From now on, the letter  $K$  will always denote a finite extension of  $K_0$ . Note that  $K$  is complete with respect to a discrete valuation  $v$ , with residue field  $k = \bar{\mathbb{F}}_p$ .

Let  $f : X \rightarrow \mathbb{P}_K^1$  be a three point cover, defined over a finite extension  $K/K_0$ . Let  $G$  denote the monodromy group of  $f$ . The theorem of Beckmann mentioned above can be reduced to the following statement: if  $p$  does not divide the order of  $G$  then  $f$  can be defined over  $K_0$ . This latter statement is, more or less, a direct consequence of Grothendieck's theory of the tame fundamental group. Indeed, if  $p$  does not divide the order of  $G$  then  $f$  extends, by Grothendieck's theory, to a tame cover  $f_{\mathcal{O}_K} : \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$ , ramified only along the sections 0, 1,  $\infty$ . Let  $\bar{f} : \bar{X} \rightarrow \mathbb{P}_k^1$  denote the special fibre of this map; it is a tame cover, ramified at most at 0, 1,  $\infty$ . By the deformation theory of tame covers, there exists a tame cover  $f_{\mathcal{O}_{K_0}} : \mathcal{X}_0 \rightarrow \mathbb{P}_{\mathcal{O}_{K_0}}^1$ , ramified at most along 0, 1,  $\infty$  lifting  $\bar{f}$ . Moreover, such a lift is unique. It follows that the generic fibre of  $f_{\mathcal{O}_{K_0}}$  is a model of  $f$  over  $K_0$ .

To see how Beckmann's Theorem follows from the above, let  $f : X \rightarrow \mathbb{P}^1$  be a three point cover, defined over  $\bar{\mathbb{Q}}$ . Let  $\mathfrak{p}$  be a place of  $\bar{\mathbb{Q}}$  whose residue characteristic is prime to the order of the monodromy group of  $f$ . Let  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  be an element of the inertia group of  $\mathfrak{p}$ . We claim that  $f^\sigma \cong f$ . Clearly, this claim implies Beckmann's Theorem.

To prove the claim, let  $K_0$  be as above. The place  $\mathfrak{p}$  gives rise to an embedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{K}_0$ . Moreover, there exists a (unique) element  $\tau \in \text{Gal}(\bar{K}_0/K_0)$  with  $\tau|_{\bar{\mathbb{Q}}} = \sigma$ . It follows from the good reduction result discussed above that the three point cover  $f_{\bar{K}_0} := f \otimes \bar{K}_0$  can be defined over  $K_0$ . Hence we have  $f_{\bar{K}_0}^\tau \cong f_{\bar{K}_0}$ , which implies  $f^\sigma \cong f$ . This proves the claim.

## 2 Stable reduction

**2.1 The stable model** Let  $K_0$  be as in §1.2, and let  $f : X \rightarrow \mathbb{P}_K^1$  be a three point cover, defined over a finite extension  $K/K_0$ . If  $p$  divides the order

of the monodromy group, then  $f$  may have bad reduction and it may not be possible to define  $f$  over  $K_0$ . For the purpose of studying this situation, it is no restriction to make the following additional assumptions.

- The cover  $f : X \rightarrow \mathbb{P}_K^1$  is Galois, with Galois group  $G$  (replace  $f$  by its Galois closure).
- The curve  $X$  has semistable reduction (replace  $K$  by a finite extension).

In the second point, we have used the Semistable Reduction Theorem, see e.g. [6]. For simplicity, we shall also assume that the genus of  $X$  is at least two. (Three point Galois covers of genus  $\leq 1$  can be classified and treated separately.) Let  $\mathcal{X}$  denote the stable model of  $X$ , i.e. the minimal semistable model of  $X$  over the ring of integers of  $K$ , see [6]. By uniqueness of the stable model, the action of  $G$  on  $X$  extends to  $\mathcal{X}$ . Let  $\mathcal{Y} := \mathcal{X}/G$  be the quotient scheme. It is shown in [14], Appendix, that  $\mathcal{Y}$  is again a semistable curve over  $\mathcal{O}_K$ .

**Definition 2.1** The morphism  $f^{\text{st}} : \mathcal{X} \rightarrow \mathcal{Y}$  is called the *stable model* of the three point cover  $f$ . Its special fibre  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  is called the *stable reduction* of  $f$ . If  $\bar{f}$  is a separable and tamely ramified map between smooth curves, then we say that  $f$  has *good reduction*. Otherwise, we say that  $f$  has *bad reduction*.

Initiated by a series of papers by Raynaud [14], [15], [16], several authors have studied the stable reduction of covers of curves (the case of three point covers is just a special case). For an overview of their results and a more extensive list of references, see [12]. In this note, we shall focus on the results of [21], and on results which inspired this work (mainly [16], [9], [20]).

**2.2 Bad reduction** Let  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  be the stable reduction of a three point cover  $f : X \rightarrow \mathbb{P}_K^1$ . Let  $(\bar{Y}_i)$  be the list of all irreducible components of the curve  $\bar{Y}$ . Since the generic fibre of  $\mathcal{Y}$  is just the projective line, the components  $\bar{Y}_i$  are all smooth curves of genus 0. Moreover, the graph of components of  $\bar{Y}$  (whose vertices are the components  $\bar{Y}_i$  and whose edges are the singular points) is a tree. For each index  $i$ , we fix an irreducible component  $\bar{X}_i$  of  $\bar{X}$  such that  $\bar{f}(\bar{X}_i) = \bar{Y}_i$ . Let  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  denote the restriction of  $\bar{f}$  to  $\bar{X}_i$ . Let  $G_i \subset G$  denote the stabilizer of the component  $\bar{X}_i$ .

The component  $\bar{Y}_i$  corresponds to a discrete valuation  $v_i$  of the function field  $K(Y)$  of  $Y = \mathbb{P}_K^1$  whose residue field is the function field of  $\bar{Y}_i$ . The choice of  $\bar{X}_i$  corresponds to the choice of a valuation  $w_i$  of the function field  $K(X)$  of  $X$  extending  $v_i$ , and the map  $\bar{f}_i$  corresponds to the residue field extension of  $w_i|v_i$ . The group  $G_i$  is simply the decomposition group of  $w_i$  in the Galois extension  $K(X)/K(Y)$ . Let  $I_i \triangleleft G_i$  denote the corresponding inertia group.

By construction, the curve  $\bar{X}$  is reduced. It follows that the ramification index of the extension of valuations  $e(w_i/v_i)$  is equal to one. This implies that the inertia group  $I_i$  is a  $p$ -group whose order is equal to the degree of inseparability of the extension of residue fields.

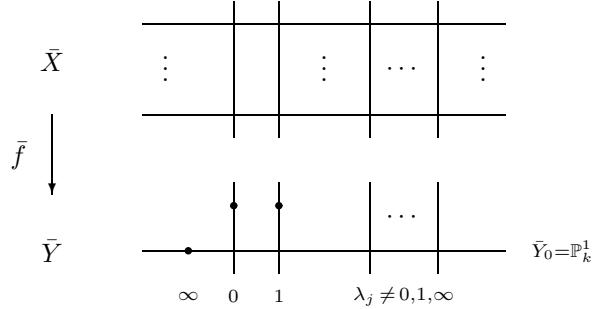


Figure 1: The stable reduction of a three point cover

We say that  $\bar{Y}_i$  is a *good component* if the map  $\bar{f}_i$  is separable. By what we have said above, this holds if and only if  $I_i = 1$ , i.e. the valuation  $v_i$  is unramified in the extension  $K(X)/K(Y)$ . If this is the case, then  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  is a Galois cover with Galois group  $G_i$ .

If  $\bar{f}_i$  is not separable we say that  $\bar{Y}_i$  is a *bad component*. The map  $\bar{f}_i$  factors as the composition of a purely inseparable map  $\bar{X}_i \rightarrow \bar{Z}_i$  of degree  $|I_i|$  and a Galois cover  $\bar{Z}_i \rightarrow \bar{X}_i$  with Galois group  $G_i/I_i$ .

Note that  $K(Y) = K(t)$ , where  $t$  is the standard parameter on  $\mathbb{P}^1$ . To simplify the exposition, we shall make the following additional assumption: there is a (necessarily unique) component  $\bar{Y}_0$  of  $\bar{Y}$  which corresponds to the Gauss valuation on  $K(t)$  with respect to the parameter  $t$ . This component is called the *original component*. It is canonically isomorphic to  $\mathbb{P}^1_k$ . (By making this assumption, we exclude the case where the cover  $f$  has bad reduction but the curve  $X$  has good reduction. In [21], Definition 2.1, this is called the *exceptional case*.) Let  $\bar{Y}_1, \dots, \bar{Y}_r$  be the components of  $\bar{Y}$  different from  $\bar{Y}_0$ .

The following theorem is the first main result of [21].

**Theorem 2.2** Suppose that  $p$  strictly divides the order of  $G$  and that  $f$  has bad reduction. Then the following holds (compare with Figure 1).

- (i) The original component  $\bar{Y}_0$  is the only bad component. Every good component  $\bar{Y}_i$  intersects  $\bar{Y}_0$  in a unique point  $\lambda_i \in \bar{Y}_0$ .
- (ii) The inertia group  $I_0$  corresponding to the bad component  $\bar{Y}_0$  is cyclic of order  $p$ . The subcover  $\bar{Z}_0 \rightarrow \bar{Y}_0$  of  $\bar{f}_0$  (which is Galois with group  $G_0/I_0$ ), is ramified at most in the points  $\lambda_i$  (where  $\bar{Y}_0$  intersects a good component).
- (iii) For  $i = 1, \dots, r$ , the Galois cover  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  is wildly ramified at the point  $\lambda_i$  and tamely ramified above  $\bar{Y}_i - \{\lambda_i\}$ . If  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  is ramified at a point  $\neq \lambda_i$ , then this point is the specialization of one of the three branch points  $0, 1, \infty$  of the cover  $f : X \rightarrow \mathbb{P}^1_K$ .

Part (ii) and (iii) of this theorem follow from part (i), by the results of [16]. In fact, this implication is not restricted to three point covers but holds for much more general covers  $f : X \rightarrow Y$ . On the other hand, the truth of part (i) depends in an essential way on the assumption that  $f$  is a three point cover.

Under the additional assumption that all the ramification indices of  $f$  are prime to  $p$ , Theorem 2.2 follows already from the results of [20], via Raynaud's construction of the *auxiliary cover* (see the introduction of [20]).

Here is a brief outline of the proof of Theorem 2.2. First, certain general results on the stable reduction of Galois covers, proved in [16], already impose severe restrictions on the map  $\bar{f}$ . For instance, it is shown that the good components are precisely the *tails* of  $\bar{Y}$  (i.e. the leaves of the tree of components of  $\bar{Y}$ ). Also, the Galois covers  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  (if  $\bar{Y}_i$  is good) and  $\bar{Z}_i \rightarrow \bar{Y}_i$  (if  $\bar{Y}_i$  is bad) are ramified at most at the points which are either singular points of the curve  $\bar{Y}$  or specialization of a branch point of  $f$ .

In the next step one defines, for each bad component  $\bar{Y}_i$ , a certain differential form  $\omega_i$  on the Galois cover  $\bar{Z}_i \rightarrow \bar{Y}_i$ . This differential form satisfies some very special conditions, relative to the map  $\bar{f} : \bar{Y} \rightarrow \bar{X}$  and the action of  $G$  on  $\bar{Y}$ . For instance,  $\omega_i$  is either logarithmic (i.e. of the form  $du/u$ ) or exact (i.e. of the form  $du$ ). Furthermore,  $\omega_i$  is an eigenvector under the action of the Galois group  $G_i/I_i$  of the cover  $\bar{Z}_i \rightarrow \bar{Y}_i$ , and its zeros and poles are related to and determined by the ramification of the map  $\bar{f} : \bar{Y} \rightarrow \bar{X}$ . These properties follow from the work of Henrio [9]. Let us say for short that  $\omega_i$  is *compatible* with  $\bar{f}$ . Intuitively,  $\omega_i$  encodes infinitesimal information about the action of the inertia group  $I_i$  on the stable model  $\mathcal{X}$ , in a neighborhood of the component  $\bar{X}_i$ . Within the proof of Theorem 2.2, the important point is that the existence of the compatible differentials  $\omega_i$  imposes further restrictions on the map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ . In fact, these restrictions are strong enough to prove part (i) of Theorem 2.2. For details, see [21], §2.1.

**2.3 Special deformation data** By Theorem 2.2 (i), the original component  $\bar{Y}_0 = \mathbb{P}_k^1$  is the only bad component for the stable reduction of the three point cover  $f$ . The proof of Theorem 2.2 shows that there exists a differential form  $\omega_0$  on the Galois cover  $\bar{Z}_0 \rightarrow \bar{Y}_0$  which is compatible with  $\bar{f}$ , in the sense explained above. It is worthwhile to write down explicitly what ‘compatibility’ implies for the differential  $\omega_0$ .

To simplify the exposition, we assume that the ramification indices of  $f$  are all divisible by  $p$ . If this is the case, then the branch points  $0, 1$  and  $\infty$  specialize to the original component  $\bar{Y}_0$ . Since we identify  $\bar{Y}_0$  with  $\mathbb{P}_k^1$ , this means that the points  $\lambda_1, \dots, \lambda_r$  where  $\bar{Y}_0$  intersects the good components  $\bar{Y}_1, \dots, \bar{Y}_r$  are distinct from  $0, 1, \infty$ . By Theorem 2.2 (iii), the Galois covers  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  are then étale over  $\bar{Y}_i - \{\lambda_i\}$ .

Let  $t$  denote the rational function on the original component  $\bar{Y}_0$  which identifies it with  $\mathbb{P}_k^1$ . Compatability of  $\omega_0$  with  $\bar{f}$  implies that

$$(1) \quad \omega_0 = c \cdot \frac{z \, dt}{t(t-1)},$$

where  $c \in k^\times$  is a constant and  $z$  is a rational function on  $\bar{Z}_0$  for which an equation of the form

$$(2) \quad z^{p-1} = \prod_i (t - \lambda_i)^{a_i}$$

holds. Here the  $a_i$  are integers  $1 < a_i < p$  such that  $\sum_i a_i = p-1$ . These integers are determined by the (wild) ramification of the Galois covers  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  at  $\lambda_i$ .

Compatibility of  $\omega_0$  with  $\bar{f}$  also implies that  $\omega_0$  is logarithmic, i.e. is of the form  $du/u$ , for some rational function  $u$  on  $\bar{Z}_0$ . Equivalently,  $\omega_0$  is invariant under the Cartier operator. The latter condition gives a finite list of equations satisfied by the  $t$ -coordinates of the points  $\lambda_i$  (depending on the numbers  $a_i$ ). One can show that these equations have only a finite number of solutions ( $\lambda_i$ ), see [19], Theorem 5.14. In other words, the existence of the differential form  $\omega_0$  determines the position of the points  $\lambda_i$ , up to a finite number of possibilities.

The pair  $(\bar{Z}_0, \omega_0)$  is called a *special deformation datum*. Given a special deformation datum  $(\bar{Z}_0, \omega_0)$ , the branch points  $\lambda_i$  of the cover  $\bar{Z}_0 \rightarrow \bar{Y}_0$  are called the *supersingular points*. A justification for this name, in form of a well known example, will be given in §4. By the result mentioned in the preceding paragraph, a special deformation datum is *rigid*, i.e. is an object with 0-dimensional moduli. This is no surprise, as special deformation data arise from three point covers, which are rigid objects themselves. We point this out, because this sort of rigidity is the (somewhat hidden) principle underlying all results discussed in this note which are particular for three point covers. In the following section, we will interpret the existence of  $(\bar{Z}_0, \omega_0)$  as a liftability condition for the map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ .

### 3 Lifting

**3.1 Special  $G$ -maps** The stable reduction of a three point Galois cover  $f : X \rightarrow \mathbb{P}_K^1$  is, by definition, a finite map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  between semistable curves over the residue field  $k$ , together with an embedding  $G \hookrightarrow \text{Aut}(\bar{X}/\bar{Y})$ . In the case of bad reduction, the curves  $\bar{Y}$  and  $\bar{X}$  are singular, and the map  $\bar{f}$  is inseparable over some of the components of  $\bar{Y}$ . This suggests the following question. Given a map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  of the sort we have just described, together with an embedding  $G \hookrightarrow \text{Aut}(\bar{X}/\bar{Y})$ , does it occur as the stable reduction of a three point Galois cover  $f : X \rightarrow \mathbb{P}_K^1$ , for some finite extension  $K/K_0$ ? If this is the case, then we say that  $f : X \rightarrow \mathbb{P}_K^1$  is a *lift* of  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ .

Theorem 2.2 and its proof give a list of necessary conditions on  $\bar{f}$  for the existence of a lift (at least under the extra condition that  $p$  strictly divides the order of  $G$ ). These conditions lead naturally to the notion of a *special  $G$ -map*. See [21], §2.2 for a precise definition. To give the general idea, it suffices to say that a special  $G$ -map is a finite map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  between semistable curves, together with an embedding  $G \hookrightarrow \text{Aut}(\bar{X}/\bar{Y})$ , which admits a compatible special deformation datum  $(\bar{Z}_0, \omega_0)$ . One can show that special  $G$ -maps are rigid in the

sense we used at the end of §2.3. Moreover, one has the following lifting result, proved in [21], §4.

**Theorem 3.1** *Let  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  be a special  $G$ -map over  $k$ . Then the following holds.*

- (i) *There exists a three point cover  $f : X \rightarrow \mathbb{P}^1$  lifting  $\bar{f}$ .*
- (ii) *Every lift  $f$  of  $\bar{f}$  can be defined over a finite extension  $K/K_0$  which is at most tamely ramified.*

The corresponding result proved in [21], §4, is somewhat stronger. It determines the set of isomorphism classes of all lifts of  $\bar{f}$ , together with the action of  $\text{Gal}(\bar{K}_0/K_0)$ , in terms of certain invariants of  $\bar{f}$  (these invariants are essentially the numbers  $a_i$  appearing in (2)). This more precise result gives an upper bound for the degree of the minimal extension  $K/K_0$  over which every lift of  $\bar{f}$  can be defined.

Theorem 1.1 follows easily from Theorem 2.2 and Theorem 3.1 (in a way similar to how Beckmann's Theorem follows from Grothendieck's theory of tame covers, see §2.1).

Part (i) of Theorem 3.1, i.e. the mere existence of a lift, follows already from the results of [20]. Part (ii) is more difficult. The technical heart of the proof is a study of the deformation theory of a certain curve with an action of a finite group scheme, which is associated to a special deformation datum. A detailed exposition of this deformation theory can be found in [19]. An overview of the proof of Theorem 3.1 will be given in §3.4 below.

**3.2 The supersingular disks** Let  $f : Y \rightarrow \mathbb{P}_K^1$  be a three point cover, defined over a finite extension  $K/K_0$ , with bad reduction. Let  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  be the stable reduction of  $f$ . We assume that the conclusion of Theorem 2.2 holds (it holds, for instance, if  $p^2 \nmid |G|$ ). Let  $Y^{\text{an}}$  denote the rigid analytic  $K$ -space associated to  $Y = \mathbb{P}_K^1$ . The  $\mathcal{O}_K$ -model  $\mathcal{Y}$  of  $Y$  yields a specialization map  $\text{sp}_{\mathcal{Y}} : Y^{\text{an}} \rightarrow \bar{Y}$ . For  $i = 1, \dots, r$ , the good component  $\bar{Y}_i$  gives rise to a rigid analytic subset

$$D_i := \text{sp}_{\mathcal{Y}}^{-1}(\bar{Y}_i - \{\lambda_i\}) \subset Y^{\text{an}}.$$

As a rigid  $K$ -space,  $D_i$  is a closed unit disk, i.e. is isomorphic to the affinoid  $\text{Spm } K\{\{T\}\}$ . See e.g. [10].

An important step in the proof of Part (ii) of Theorem 3.1 is to show that the disks  $D_i$  depend only on the reduction  $\bar{f}$ , but not on the lift  $f$  of  $\bar{f}$ . For simplicity, we shall again assume that all the ramification indices of the three point cover  $f : X \rightarrow \mathbb{P}_K^1$  are divisible by  $p$ , see §2.3. Then the special deformation datum  $(\bar{Z}_0, \omega_0)$  associated to the reduction  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  is essentially determined by points  $\lambda_1, \dots, \lambda_r \in \mathbb{P}_k^1 - \{0, 1, \infty\}$  and integers  $a_1, \dots, a_r$  with  $1 < a_i < p$  and  $\sum_i a_i = p - 1$ . We consider  $\lambda_i$  as an element of  $k - \{0, 1\}$  and let  $\tilde{\lambda}_i \in \mathcal{O}_{K_0}$  be a lift of  $\lambda_i$ . By definition, the closed disk  $D_i$  is contained in the open unit disk

$$D'_i := \{t \in \mathcal{O}_K \mid |t - \tilde{\lambda}_i|_K < 1\} \subset Y^{\text{an}}.$$

With this notation, we have the following result, see [21], Proposition 4.3 and the remark following the proof of Theorem 3.8.

**Proposition 3.2** We have

$$D_i = \{ t \in D'_i \mid |t - \tilde{\lambda}_i|_K \leq |p|_K^{\frac{p}{p-1+a_i}} \}.$$

In particular, the disks  $D_i$  depend only on the special deformation datum  $(\bar{Z}_0, \omega_0)$ .

We call the open disks  $D'_i$  the *supersingular disks* associated to the special deformation datum  $(\bar{Z}_0, \omega_0)$ . This is in correspondence with naming the points  $\lambda_i \in \bar{Y}_0$  the supersingular points, see §2.3. The closed subdisk  $D_i \subset D'_i$  is called the *too supersingular disk*, a term which is also borrowed from the theory of moduli of elliptic curves, see §4.

**3.3 The auxiliary cover** We continue with the notation and assumptions of the preceding subsection. Recall that we have chosen in §2.2 a component  $\bar{X}_i \subset \bar{X}$  above the component  $\bar{Y}_i \subset \bar{Y}$ . The stabilizer of  $\bar{X}_i$  is the subgroup  $G_i \subset G$  and the inertia subgroup of  $\bar{X}_i$  is a normal subgroup  $I_i \triangleleft G_i$ . By the conclusion of Theorem 2.2, we have  $|I_0| = p$  and  $|I_i| = 1$  for  $i = 1, \dots, r$ . Let  $\bar{X}^{\text{hor}} \subset \bar{X}$  denote the union of all components mapping to the original component  $\bar{Y}_0$ . Then we have

$$f^{-1}(D_i) = \text{Ind}_{G_i}^G(E_i), \quad \text{with } E_i = \text{sp}_{\mathcal{X}}^{-1}(\bar{X}_i - \bar{X}^{\text{hor}}) \subset X^{\text{an}}.$$

The map  $E_i \rightarrow D_i$  is a finite étale Galois cover between smooth affinoid  $K$ -spaces whose reduction is equal to the restriction of the étale Galois cover  $\bar{f}_i^{-1}(\bar{Y}_i - \{\lambda_i\}) \rightarrow \bar{Y}_i - \{\lambda_i\}$ . This determines  $E_i \rightarrow D_i$  uniquely, up to isomorphism, because lifting of étale morphisms is unique.

Let  $U_0 := Y^{\text{an}} - (\cup_i D_i)$  denote the complement of the disks  $D_i$ . Then we have

$$f^{-1}(U_0) = \text{Ind}_{G_0}^G(V_0), \quad \text{with } V_0 := \text{sp}_{\mathcal{X}}^{-1}(\bar{X}^{\text{hor}}) \subset X^{\text{an}}.$$

The map  $V_0 \rightarrow U_0$  is a finite Galois cover between smooth (non quasicompact) rigid  $K$ -spaces, étale outside the subset  $\{0, 1, \infty\} \subset U_0$ . It can be shown that there exists a  $G_0$ -Galois cover  $f^{\text{aux}} : X^{\text{aux}} \rightarrow Y = \mathbb{P}_K^1$  such that  $V_0 = (f^{\text{aux}})^{-1}(U_0)$ . Such a cover  $f^{\text{aux}}$  is ramified at  $0, 1, \infty$  and, for each  $i = 1, \dots, r$ , at one point  $y_i \in D'_i$ . Moreover, the cover  $f^{\text{aux}}$  is uniquely determined by the choice of the points  $y_i$ . It is called the *auxiliary cover* associated to  $f$  and the points  $(y_i)$ , see [16], [20] and [21], §4.1.3. Let  $\partial D_i$  denote the boundary of the disk  $D_i$ . By construction of  $f^{\text{aux}}$ , there exists a  $G$ -equivariant isomorphism

$$(3) \quad \varphi_i : \text{Ind}_{G_i}^G(E_i \times_{D_i} \partial D_i) \cong \text{Ind}_{G_0}^G(f^{\text{aux}})^{-1}(\partial D_i),$$

compatible with the natural map to  $\partial D_i$ .

**3.4 The proof of Theorem 3.1** We will now give a brief outline of the proof of Theorem 3.1. Suppose that  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  is a special  $G$ -map. We want to construct all three point covers  $f : X \rightarrow Y = \mathbb{P}_K^1$  lifting  $\bar{f}$ . For the moment, we let  $K$  be any sufficiently large finite extension of  $K_0$ . At the end of our argument, we will reason that it suffices to take for  $K$  a certain tame extension of  $K_0$ , which is explicitly determined by  $\bar{f}$ .

We divide the proof into three steps. The first step consists in constructing a Galois cover  $f^{\text{aux}} : X^{\text{aux}} \rightarrow Y$  which can play the role of the auxiliary cover associated to any lift  $f$  of  $\bar{f}$ . In fact, one shows that all good candidates  $f^{\text{aux}}$  live in a continuous family which depends only on the special deformation datum  $(\bar{Z}_0, \omega_0)$  associated to  $\bar{f}$ . The individual members of this family depend on the choice of the extra branch points  $y_i \in D'_i$  (with  $D'_i$  as above). See [20], §3.2, and [21], §3.

Let  $D_i \subset D'_i$  be a smaller closed disk, defined as in Proposition 3.2 (the numbers  $a_i$  used in the definition of  $D_i$  are determined by the special deformation datum  $(\bar{Z}_0, \omega_0)$ ). Let  $E_i \rightarrow D_i$  be the étale  $G_i$ -Galois cover lifting  $\bar{f}_i^{-1}(\bar{Y}_i - \{\lambda_i\}) \rightarrow \bar{Y}_i - \{\lambda_i\}$ . For any choice of points  $y_i \in D'_i$ , we obtain a cover  $f^{\text{aux}} : X^{\text{aux}} \rightarrow Y$ , which is a candidate for the auxiliary cover. In this situation, a tuple  $(\varphi_i)$  of isomorphisms as in (3) is called a *patching datum*.

The second step of the proof consists in showing that there exists a patching datum  $(\varphi_i)$  if and only if the points  $y_i$  lie in the smaller disk  $D_i$ . The sufficiency of the condition  $y_i \in D_i$  can be shown using the same arguments as in [20], §3.4. The necessity of this condition – which is equivalent to Proposition 3.2 above – lies somewhat deeper. See [21], §3, in particular Theorem 3.8. In this step one uses the deformation theory developed in [19].

The third and final step uses rigid (or formal) patching. For any choice of  $y_i \in D_i$ , let  $f^{\text{aux}} : X^{\text{aux}} \rightarrow Y = \mathbb{P}_K^1$  be the associated auxiliary cover, and set  $V_0 := (f^{\text{aux}})^{-1}(U_0)$ . By the second step, we have a patching datum  $(\varphi_i)$ . The proof of the claim in step two shows moreover that the cover  $V_0 \rightarrow U_0$  depends only on the special deformation datum, but not on the choice of  $y_i \in D_i$ . Using rigid patching, one easily constructs a  $G$ -Galois cover  $f : X \rightarrow Y$  such that  $f^{-1}(D_i) = \text{Ind}_{G_i}^G(E_i)$ ,  $f^{-1}(U_0) = \text{Ind}_{G_0}^G(V_0)$  and such that the patching datum  $(\varphi_i)$  is induced by the identity on  $X$ . Essentially by construction,  $f$  is a three point cover lifting the special  $G$ -map  $\bar{f}$ . This proves Part (i) of Theorem 3.1.

It is not hard to see that all lifts of  $\bar{f}$  arise in the way we have just described. More precisely, the set of isomorphism classes of lifts of  $f$  is in bijection with the set of patching data. Therefore, to finish the proof of Theorem 3.1 it suffices to show that the above construction works over a tame extension  $K/K_0$ . Actually, the construction of the covers  $E_i \rightarrow D_i$  and of the auxiliary cover  $f^{\text{aux}} : X^{\text{aux}} \rightarrow Y$  can be done over  $K_0$  (set  $y_i := \bar{\lambda}_i$ ). A direct analysis shows that patching data  $(\varphi_i)$  exist if one takes for  $K$  the (unique) tame extension of degree  $(p-1) \cdot \text{lcm}_i(p-1+a_i)$ . This concludes the proof of Theorem 3.1.

## 4 Modular curves

**4.1 Modular curves as three point covers** Let  $\mathbb{H}$  denote the upper half plane and  $\mathbb{H}^*$  the union of  $\mathbb{H}$  with the set  $\mathbb{P}^1(\mathbb{Q})$ . The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathbb{H}$  and  $\mathbb{H}^*$  in a standard way. Moreover, for every subgroup  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$  of finite index, the quotient  $X_\Gamma := \mathbb{H}^*/\Gamma$  carries a natural structure of a compact Riemann surface, and therefore also of a smooth projective curve over  $\mathbb{C}$ . The classical  $j$ -function identifies the quotient of  $\mathbb{H}^*$  by  $\mathrm{SL}_2(\mathbb{Z})$  with the projective line. So for each finite index subgroup  $\Gamma$  we obtain a finite cover of compact Riemann surfaces (or smooth projective curves over  $\mathbb{C}$ )

$$f_\Gamma : X_\Gamma \longrightarrow \mathbb{P}^1.$$

This map is unramified away from the three points  $0, 1728, \infty$ . In other words,  $f_\Gamma$  is a three point cover.

For an integer  $N$ , let

$$\Gamma(N) := \{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv I_2 \pmod{N} \}.$$

A *congruence subgroup* is a subgroup  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$  which contains  $\Gamma(N)$ , for some  $N$ . The corresponding curve  $X_\Gamma$  is called a *modular curve*. The standard examples for congruence subgroups are  $\Gamma(N)$ ,  $\Gamma_0(N)$  and  $\Gamma_1(N)$ . The corresponding modular curves are usually denoted by  $X(N)$ ,  $X_0(N)$  and  $X_1(N)$ .

**4.2 The modular curve  $X(p)$**  Let us fix an odd prime number  $p$ . The three point cover  $g : X(p) \rightarrow X(1) = \mathbb{P}^1$  is Galois, with Galois group  $G = \mathrm{PSL}_2(p)$ . Its ramification index at the branch point  $\infty$  (resp. at  $0$ , resp. at  $1728$ ) is equal to  $p$  (resp.  $3$ , resp.  $2$ ). Note that the order of  $G$  is equal to  $p(p^2 - 1)/2$ .

For technical reasons it is easier to discuss a variant of  $g$ , namely the three point cover

$$f : X(2p) \rightarrow X(2) = \mathbb{P}^1.$$

(We identify the modular curve  $X(2)$  with  $\mathbb{P}^1$  by means of the classical  $\lambda$ -function.) The cover  $f$  is Galois with Galois group  $G = \mathrm{PSL}_2(p)$ . The ramification index at each of the three branch points  $0, 1, \infty$  is equal to  $p$ . There is an equivariant action of  $\mathrm{SL}_2(2) \cong S_3$  on the source and the target of  $f$ . The cover  $g$  is obtained by taking the quotient under this action.

The next proposition follows easily from the results of [7].

**Proposition 4.1** *The covers  $f$  and  $g$  can be defined over  $\mathbb{Q}$ . In particular,  $\mathbb{Q}$  is their field of moduli. Furthermore,  $g$  (resp.  $f$ ) has good reduction at the prime  $l$  (in the sense of Definition 2.1) for  $l \neq 2, 3, p$  (resp. for  $l \neq p$ ).*

The fact that  $f$  and  $g$  can be defined over  $\mathbb{Q}$  follows also from the *rigidity* criterion used in inverse Galois theory. See [13] or [18]. The  $\mathbb{Q}$ -models of  $f$  and  $g$  are not unique. However, both covers have a unique model over the field  $\mathbb{Q}(\sqrt{p^*})$  which is Galois.

Note that the second statement of Proposition 4.1 confirms (but is not implied by) the good reduction criterion discussed in §1.2. For  $l = 2, 3$ , the curve  $X(p)$  has good reduction as well. However, since in this case  $l$  divides one of the ramification indices, the cover  $g$  has bad reduction – at least in the sense of Definition 2.1.

Since  $p$  exactly divides the order of  $G$ , the results discussed in §2 and §3 can be used to study the stable reduction of  $f$  and  $g$  at the prime  $p$ . This is done in detail in [3]. We shall present some of the main results of [3], as an illustration for the results discussed earlier.

Let  $K_0$  be the completion of the maximal unramified extension of  $\mathbb{Q}_p$ . From now on, we consider the three points covers  $f$  and  $g$  as defined over  $K_0$ . We remark that there are many different models of these covers over  $K_0$ . However, the stable model, which exists over a finite extension of  $K_0$ , is unique.

For simplicity, we discuss only the stable reduction of the cover  $f$  in detail. Let  $K/K_0$  be the minimal extension over which  $f$  has semistable reduction, and let  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  denote the stable reduction of  $f$ . We will freely use the notation introduced in §2.2 and §2.3. Since  $f$  has bad reduction and  $p$  strictly divides the order of  $G$ , the conclusion of Theorem 2.2 holds. In particular,  $\bar{f}$  gives rise to a special deformation datum  $(\bar{Z}_0, \omega_0)$  and to Galois covers  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$ . Since all the ramification indices of the cover  $f$  are equal to  $p$ , none of the supersingular points  $\lambda_i \in \bar{Y}_0 = \mathbb{P}_k^1$  equal 0, 1 or  $\infty$ . Moreover, the Galois covers  $\bar{f}_i$  are étale over  $\bar{Y}_i - \{\lambda_i\}$ .

#### Theorem 4.2

- (i) The field  $K$  is the (unique) tame extension of  $K_0$  of degree  $(p^2 - 1)/2$ .
- (ii) There are  $r = (p - 1)/2$  supersingular points  $\lambda_i$ ; they are precisely the roots of the Hasse polynomial

$$\Phi(t) = \sum_{j=0}^r \binom{r}{j}^2 t^j.$$

The Galois cover  $\bar{Z}_0 \rightarrow \bar{Y}_0$  is the cyclic cover of degree  $r$  given by the equation  $z^r = \Phi(t)$ . Furthermore, we have  $\omega_0 = z t^{-1} (t - 1)^{-1} dt$ .

- (iii) For  $i = 1, \dots, r$ , the curve  $\bar{X}_i$  is given by the equation  $y^{(p+1)/2} = x^p - x$ . An element of  $G = \mathrm{PSL}_2(p)$ , represented by a matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(p)$ , acts on  $\bar{X}_i$  as follows:

$$A(x) = \frac{ax + b}{cx + d}, \quad A(y) = \frac{y}{(cx + d)^2}.$$

For the proof of this theorem, see [3]. The main idea is this. One constructs a special  $G$ -map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ , which satisfies the conclusion of Theorem 4.2 (ii) and (iii). By Theorem 3.1, it lifts to a three point cover  $f' : X' \rightarrow \mathbb{P}^1$ , defined

over the tame extension of  $K_0$  of degree  $(p^2 - 1)/2$ . By the rigidity criterion of inverse Galois theory,  $f'$  has to be isomorphic to  $f$ . Whence the theorem.

One can prove a similar theorem about the stable reduction of  $g : X(p) \rightarrow X(1) = \mathbb{P}^1$ . If  $p \equiv 1 \pmod{12}$  then the statements are almost identical, except that the Hasse polynomial has to be replaced by another polynomial (for which there is an explicit formula, similar to the expression for  $\Phi$ ). If  $p \not\equiv 1 \pmod{12}$  then some of the supersingular points  $\lambda_i$  are equal to either 0 or 1728, and the corresponding Galois cover  $\bar{f}_i : \bar{X}_i \rightarrow \bar{Y}_i$  is not the one from Theorem 4.2 (iii).

The modular curves  $X_0(p)$ ,  $X_1(p)$  and  $X_0(p^2)$  are all quotients of the curve  $X(p)$ . Using the results of [3] on the stable reduction of  $X(p)$ , one can determine the stable reduction of all these quotients, reproving results of Deligne–Rapoport [7] and Edixhoven [8]. Somewhat surprisingly, this new proof does not use the interpretation of modular curves as moduli spaces for elliptic curves with level structure.

However, the ‘modular’ interpretation of modular curve justifies the use of the term ‘supersingular’ in §2.3 and §3.2. In fact, the supersingular points  $\lambda_i$ , which are the roots of the Hasse polynomial  $\Phi$ , are exactly the values  $t \in k$  for which the Legendre elliptic curve  $E_t$  with equation  $y^2 = x(x-1)(x-t)$  is supersingular. Similarly, a point  $t \in \mathbb{P}^1(K)$  lies in one of the open disks  $D'_i$  if and only if the elliptic curve  $E_t$  has supersingular reduction. The modular interpretation of the smaller closed disks  $D_i \subset D'_i$  is somewhat less known. However, it can be shown (see e.g. [4]) that a point  $t \in \mathbb{P}^1(K)$  lies in one of the disks  $D_i$  if and only if the elliptic curve  $E_t$  is *too supersingular*, which means that  $E_t[p]$  has no canonical subgroup.

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